Polynomials

Polynomials are functions of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer (≥ 0), and $a_n, a_{n-1}, \dots, a_2, a_1, a_0$ are real numbers $(\in \mathbb{R})$.

a_n is called "leading coefficient"

a_0 is called "constant term".

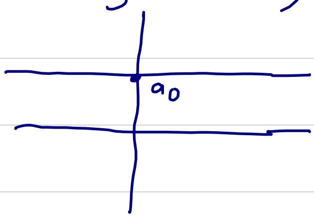
Also to avoid degenerate cases we will assume $a_n \neq 0$.

Cases. (i) $n = 0$

$$f(x) = a_0$$

This is just the constant function.

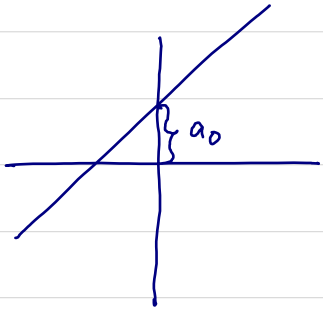
(horizontal line)



(ii) $n = 1$

$$f(x) = a_1 x + a_0$$

This is the linear function which we know so well. a_0 is y-intercept and a_1 is slope.



(line)

(iii) $n = 2$

$$f(x) = a_2 x^2 + a_1 x + a_0$$

This is the quadratic function which I assume you know well from previous courses. The graph is a parabola.

(parabola)

Q. Why study polynomials?

A. They are very simple functions. Although they do not consist of all functions you can think of, you will learn in higher mathematics that you can approximate continuous functions using polynomials. In other words for every continuous function, there is a polynomial which looks almost the same. Calculus for polynomials is also very simple. Polynomials are the building blocks for the set of continuous functions.

In this section we will focus on quadratic functions, i.e. the case when $n=2$.

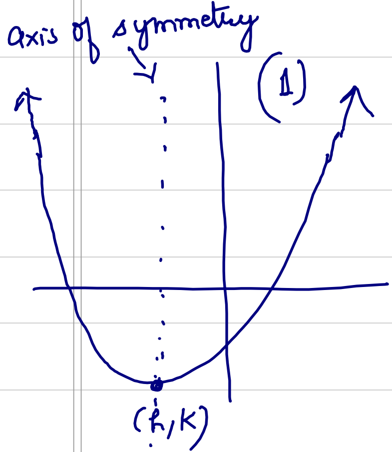
We will replace the nasty a_2, a_1, a_0 by simpler a, b and c. So quadratic functions are functions of the form

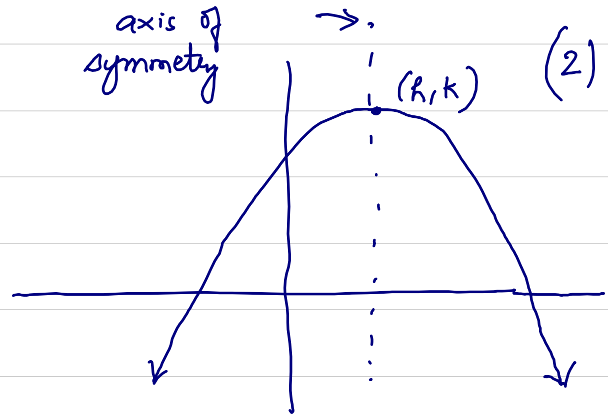
$$f(x) = ax^2 + bx + c.$$

where $a, b, c \in \mathbb{R}$ and $a \neq 0$.

Q. What does the graph of a quadratic look like?

A. The graph is a parabola that either opens upward or opens downward.





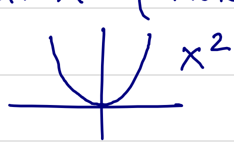
A quadratic has a vertex denoted as (h, k). When the parabola opens upward (1) the vertex is a minimum point, i.e. the function has a minimum value and it is equal to k. This is because the graph does not go below the vertex. Similarly, when the parabola opens downward (see fig (2)) the vertex is a maximum point, i.e. the function has a maximum value and it is equal to k.

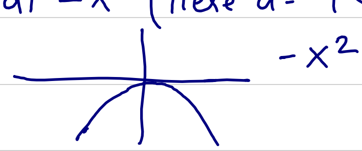
A parabola is symmetric with respect to the vertical line passing through the vertex (h, k). This vertical line is called the "axis of symmetry."

Q. How to know whether the graph opens upward or downward?

A. For $f(x) = ax^2+bx+c$ if $a>0$, then it opens upward. If $a<0$ then it opens downward.

Mnemonic: look at x^2 (Here $a=1>0$) look at $-x^2$ (Here $a=-1<0$)





The quadratic function written as

$$f(x) = ax^2 + bx + c \qquad (a \neq 0)$$

is the general form

There are three ways to write the same quadratic function:

1) General form : $ax^2 + bx + c$
2) Standard form (Also called the Vertex form) : $a(x-h)^2 + K$
3) Factored form : $a(x-r)(x-s)$

Recall that when we looked at the linear function $f(x) = mx + b$, the coefficients gave us simple information about the graph of the function, namely the slope and y-intercept of the line.

Similarly for $f(x) = ax^2 + bx + c$, we would naturally expect the coefficients a, b, and c to give us simple, graphical information about the graph of the parabola. So for example, if someone gives me a nasty function like

$$f(x) = 0.799\,x^2 - \frac{1}{4}x + 1005$$

I can analyze the graph without plotting a bunch of points.

We already know one information: if $a>0$ then it opens upward and if $a<0$ then it opens downward. What about the vertex?

That is why we need to know the standard/vertex form.

Standard form

This is given by

$$f(x) = a(x-h)^2 + k$$

h gives the x-coordinate of the vertex and k gives the y-coordinate of the vertex. The a is the same leading coefficient as in the general form.

Ques How is this a quadratic function?

Ans. We have

$$\begin{aligned} f(x) &= a(x-h)^2 + k \\ &= a(x^2 - 2xh + h^2) + k \\ &= \underbrace{a}x^2 \underbrace{- 2ah}x + \underbrace{ah^2 + k} \end{aligned}$$

Remember

$(a+b)^2$
$= a^2+2ab+b^2$

$(a-b)^2$
$= a^2-2ab+b^2$

a^2-b^2
$= (a+b)(a-b)$

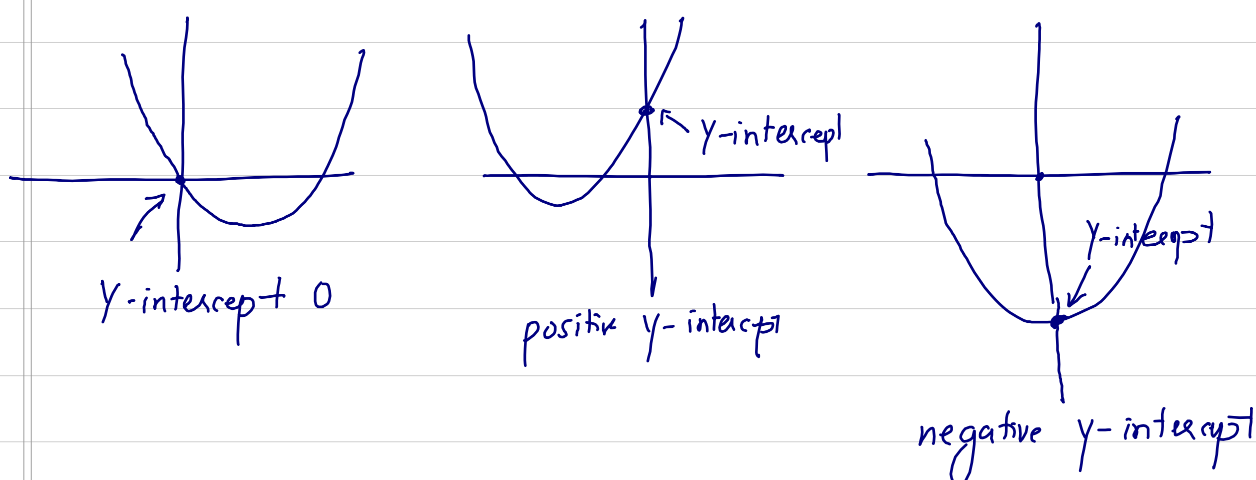
Note a, $-2ah$, ah^2+k are constant real numbers. Therefore, the standard form represents a quadratic function. □

Ques Can every quadratic function in general form written as a quadratic in vertex form?

Ans. Yes. This is discussed at the end.

X- intercepts and Y-intercepts of a quadratic function

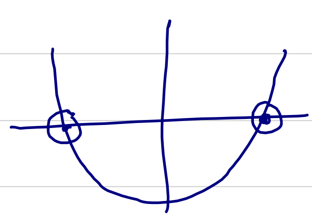
The Y- intercept is the point where the quadratic function intersects the Y-axis. A quadratic will always intersect the Y-axis:



The X-intercept is the ______ point where the quadratic intersects the X-axis.

Q. What is the value of f at a point of X-intercept?

Ans.



At the point of X-intercept, the value of f is 0, i.e. for x if x is an X-intercept, then $f(x) = 0$.

Thus, $\{\text{X-intercepts}\} = \{x \text{ such that } f(x) = 0\}$.

In other words, finding the x-intercepts is the same as solving the equation $f(x) = 0$.

Q. Is there always an x-intercept?

Ans. This question can be formulated as "Is there always a solution for $f(x)=0$ where f is a quadratic.

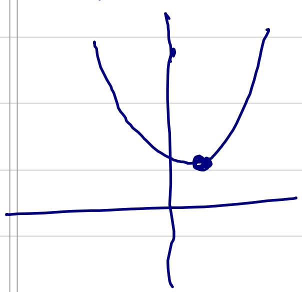
The answer is no. A quadratic equation can have either (i) no real solutions.

(ii) one real solution (with multiplicity 2)

(iii) two real solutions.

Later in the chapter we will see that when we allow for complex solutions, the theory becomes the most beautiful theory one could imagine. Namely, there will always be 2 solutions allowing for repeated solutions.

Graphical answer



No real solution. Because the minimum point is above the X-axis

No real solution. Because the maximum point is below the X-axis.

How to graph a quadratic function given in standard form:

$$f(x) = a(x-h)^2 + k.$$

Step 1) Check $a>0$ or $a<0$. This tells us if

Step 2) Determine the vertex (h,k). This is simple
This tells us the coordinate of the vertex.

Step 3) Determine the y-intercept. This tells us the point where the parabola intersects the y-axis.
How to find it? Just plug in $x=0$. Then that will give you the y-intercept.

Step 4) Determine the x-intercept. This tells you the points where the parabola intesects the x-axis.
You just need to solve $f(x)=0$.

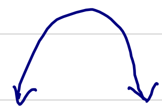
Step 5) Plot vertex
y-intercept
x-intercept
and connect the dots.

Example 1

Graph $f(x) = -3(x+1)^2 - 2$.

Solution.

Step 1. $a = -3$ which is negative. Thus,



Step 2. $f(x) = -3(x+1)^2 - 2$

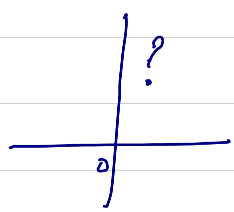
$= -3(x-(-1))^2 + (-2)$. Compare this to

$$\boxed{f(x) = a(x-h)^2 + k}$$

we get $h = -1$, $k = -2$.

⚠ People tend to make mistake in this step. For example they write $h = 1$ and $K = 2$ or -2.

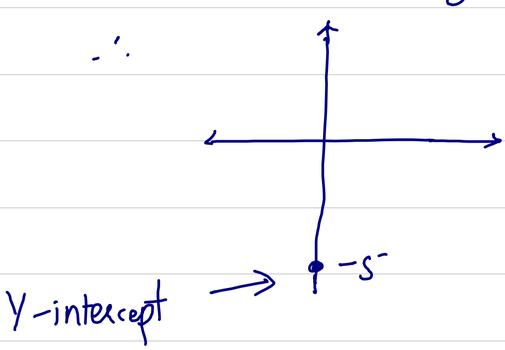
Step 3 Y-intercept.



We just need to plug $x = 0$. Then

$$f(0) = -3(0+1)^2 - 2$$
$$= -3 - 2$$
$$= -5$$

$\therefore$



Step 4 X-intercept

We need to solve $f(x) = 0$,

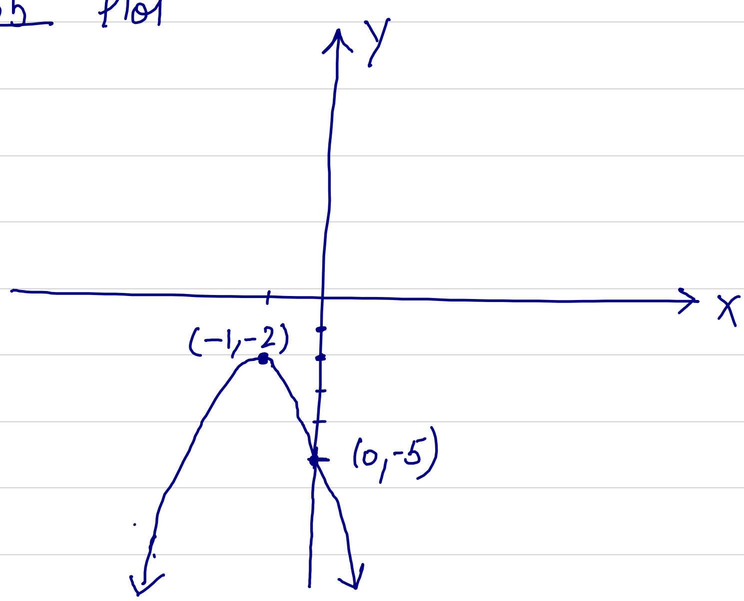
$$\Rightarrow \quad -3(x+1)^2 - 2 = 0$$

$$\Rightarrow \quad -3(x+1)^2 = 2$$

$$\Rightarrow \quad (x+1)^2 = -\frac{2}{3}$$

A square on the left hand side cannot be negative. So there are no real solutions or X- intercepts.

Step 5 Plot



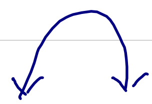
Exercise: Graph $f(x) = (x-1)^2 - 4$

Q. What if I am told to graph $f(x) = -3x^2 + 6x + 2$?

Ans. $f(x) = -3x^2 + 6x + 2$

Let's see what we can information we can scavange and then we will think about the next step.

✓ We know a (leading coefficient): $a = -3$.

So graph = 

? Vertex? No Idea.

✓ Y-intercept? Yes. $f(0) = -3 \cdot 0^2 + 6 \cdot 0 + 2$
$= 2$.

(Postponed) X-intercept? We can do this now using quadratic formula but it will be extra work. We can worry about this later

Vertex is what we want.

Completing the square

Memorize the formulas: $\begin{cases} (a+b)^2 = a^2 + 2ab + b^2 \\ (a-b)^2 = a^2 - 2ab + b^2 \end{cases}$

This is important. Write it ≥ 10 times.

We have $f(x) = -3x^2 + 6x + 12$

x^2 will always be the a^2.

This will give rise to $2ab$ term

We have to produce the b^2 term out of nowhere.

$$f(x) = -3x^2 + 6x + 2$$

We don't want this pesky leading coefficient. Factor it out from the first two terms.

$$= -3(x^2 - 2x) + 2$$

Now we have a^2. Thus $a = x$

This must be $-2ab$

$$= -3(x^2 - 2x \cdot 1) + 2$$

So now we know what b must equal. $b = 1$. Now remember $(a-b)^2 = a^2 - 2ab + b^2$. We are missing the b^2 term. Just add and subtract b^2.

$$= -3(\underbrace{x^2 - 2 \cdot x \cdot 1 + 1^2} - 1^2) + 2$$

$$= -3\left((x-1)^2 - 1\right) + 2$$

$$= -3(x-1)^2 + 3 + 2$$

$$= -3(x-1)^2 + 5.$$

Now we know the vertex. It is $(1, 5)$.

Let us calculate x-intercepts:

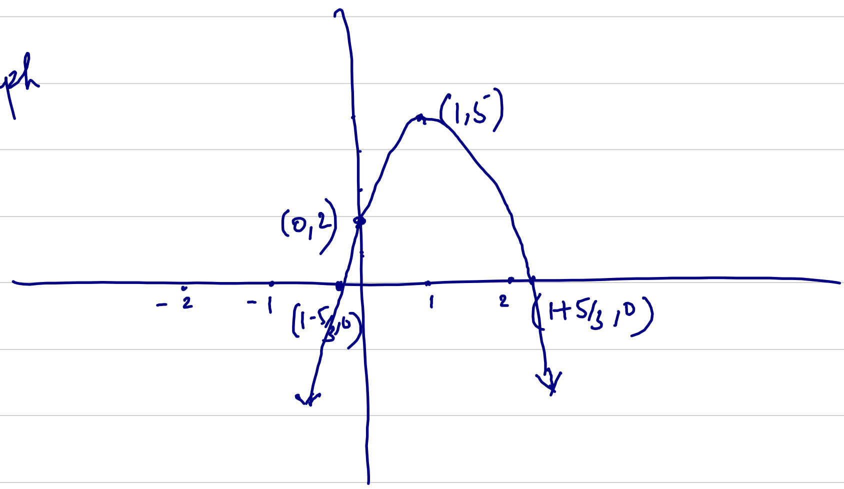
$$f(x) = 0$$

$$-3(x-1)^2 + 5 = 0$$

$$(x-1)^2 = \frac{5}{3}$$

$$x - 1 = \pm 5/3 \Rightarrow x = 1 \pm \frac{5}{3}$$

Graph



Exercise:

Complete the square.

(i) $f(x) = \underset{\substack{\uparrow \\ a^2}}{x^2} + \underset{\substack{\uparrow \\ 2ab}}{10x}$

[Remember: $(a+b)^2 = a^2 + 2ab + b^2$]

$\therefore f(x) = x^2 + 2 \cdot x \cdot 5 + 5^2 - 5^2$
$= (x+5)^2 - 25$

(ii) $f(x) = -5x^2 + 100x - 36$
$= -5(x^2 - 20x) - 36$
$= -5(x^2 - 2 \cdot x \cdot 10 + 10^2 - 10^2) - 36$
$= -5\{(x-10)^2 - 100\} - 36$
$= -5(x-10)^2 + 500 - 36$
$= -5(x-10)^2 + 464$

Exercise:

Complete the square:

$f(x) = -\frac{1}{3}x^2 + \frac{2}{5}x + 4$